MFIEs Applied to Toroidal Structures Have Spurious Solutions

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Abstract

In the presence of toroidal surfaces, the MFIE has a non-trivial nullspace in the static limit. Here, this is proven by explicitly constructing a basis for this nullspace. Next, the effects of this nullspace on the numerical solution of both the frequency domain and the time-domain MFIE's are investigated. Numerical examples are given that corroborate these statements.

1. Introduction

Boundary integral equations are often used to simulate scattering by closed perfect electrically conductors (PEC). Among the many available alternatives, the electric and magnetic field integral equation (EFIE and MFIE) are the most popular; both can be formulated in the frequency and time domains. The EFIE is more versatile than the MFIE: it applies to open structures and wires, and is easily modified to account for surface resistances and impedances. Moreover, the EFIE typically is more accurate than the MFIE. That said, there are situations were the MFIE is the more sensible choice. Indeed, the linear systems resulting from discretization of the MFIE generally are better conditioned than those resulting from discretization of the EFIE. This is because the MFIE is an equation of the second kind whereas the EFIE is an equation of the first kind. The spectrum of the former is bounded and has a finite non-zero accumulation point, while the latter has a spectrum accumulating both at zero and infinity. This does not mean that the MFIE is devoid of any spectrum related problems. In the moderate to high frequency regime, the MFIE is ill-posed at frequencies where the cavity formed by the scatterer supports resonant modes. The traces of these resonant fields reside in the nullspace of the MFIE operator. After discretization, the existence of these resonant fields results in ill-conditioned systems. In the low frequency regime, however, the MFIE generally is free from resonances when applied to simply connected geometries. However, applied to non-trivial topologies the picture becomes more complicated. Indeed, as will be shown in this contribution, the static MFIE operator has a nullspace when applied to multiply connected geometries. And although it makes no sense to use the MFIE in the static regime, the presence of this nullspace dramatically affects quasi-static simulations. In this paper, the construction of a basis for the nullspace of the static MFIE operator will be sketched and the effect of its existence on the MFIE-based simulation of non-static frequency domain and time-domain problems will be elucidated.

2. Integral Equations and Discretization

Since this contribution focuses on the effects of a non-trivial nullspace of the MFIE operator on both frequency domain and time-domain computations, some notational conventions are introduced to distinguish both cases. Transient currents and fields are represented using bold upper case Roman symbols $(\boldsymbol{H}(r,t), \boldsymbol{J}(r,t))$. Their frequency domain counterparts are denoted by bold lower case symbols ($h(r), j(r)$); their frequency dependence is suppressed. Time-domain operators are denoted by calligraphic symbols ($\mathcal{K}$) and frequency domain operators by Roman capitals (K). In discretized form, frequency domain quantities are recognized by the absence of a temporal subscript. Frequency domain and time-domain MFIE's are disambiguated by including the prefixes FD and TD. All transient signals are assumed causal (i.e. they vanish for $t < 0$).

Consider a closed PEC scatterer with boundary Γ and exterior normal $\hat{n}$, which is illuminated by an incident wave $H^i(r,t)$ or $h^i(r)$. Enforcing the magnetic boundary condition on Γ yields the FD-MFIE

$$\hat{n} \times h^i(r) = \left\{ \frac{1}{2} + K \right\} [j(r)] = \frac{j(r)}{2} - \hat{n} \times \int_\Gamma dS' \nabla \times \frac{e^{-jkR}}{4\pi R} j(r') \tag{1}$$

and the TD-MFIE

$$\hat{n} \times H^i(r,t) = \left\{ \frac{1}{2} + \mathcal{K} \right\} [J(r,t)] = \frac{J(r,t)}{2} - \hat{n} \times \int_\Gamma dS' \nabla \times \frac{J(r', t - R/c)}{4\pi R}, \tag{2}$$

for all $r \in \Gamma$ and $t > 0$. Schemes for discretizing these equations are described in [1, 2], and yield the FD-MFIE method of moments (MOM) system

$$\left(\frac{1}{2} \mathbf{I} + \mathbf{K} \right) \cdot \mathsf{J} = \mathsf{H}^i \tag{3}$$

and the TD-MFIE marching-on-in-time (MOT) system

$$\frac{1}{2} \mathbf{I} \cdot \mathsf{J}_j + \sum_{k=0}^{k_{\max}} \mathbf{K}_k \cdot \mathsf{J}_{j-k} = \mathsf{H}^i_j. \tag{4}$$

The latter system can be solved for all J_j, starting with J_0. Both systems typically are solved iteratively. The condition number of the matrices that need to be inverted (i.e. $\mathbf{K} + \mathbf{I}/2$ and $\mathbf{K}_0 + \mathbf{I}/2$) is therefore of utmost importance. Moreover, the MOT system can be unstable; this means that the excitation will couple to spurious non-decaying regime solutions that pollute the physical solution. This effect is most pronounced after the forcing term has decayed.

3. Nullspace of the static MFIE operator

In the zero frequency limit, the FD-MFIE becomes

$$\hat{n} \times h^i(r) = \left\{ \frac{1}{2} + K^s \right\} [j(r)] = \frac{j(r)}{2} - \hat{n} \times \int_\Gamma dS' \nabla \times \frac{1}{4\pi R} j(r'). \tag{5}$$

In the case where Γ is an N-torus, i.e. a generalized torus with N holes, the static MFIE operator $1/2 + K^s$ has a non-trivial nullspace comprising N linear independent current configurations. The construction of these nullspace elements will now be sketched. This construction follows the theory of [3, 4], applied to the MFIE operator. Denote the interior of the PEC by Ω^- and its exterior by Ω^+. Now let L_i be a closed loop in Ω^-, circling only hole i (once) and denote by $\hat{l}$ the tangential unit vector along this loop. The magnetic field caused in Ω^+ by a current of unit amplitude running along L_i, in the absence of the PEC is

$$h_{i,0}(r) = \int_{L_i} dl' \nabla \frac{1}{4\pi R} \times \hat{l}' dl'. \tag{6}$$

A second contribution is defined by $h_{i,1}(r) = \nabla \psi(r)$ where $\psi(r)$ is the unique bounded solution in Ω^+ of

$$\nabla^2 \psi = 0, \quad \frac{\partial \psi}{\partial n} = -\hat{n} \cdot h_{i,0}. \tag{7}$$

Since $\int_\Gamma dS' \hat{n} \cdot h_{i,0}(r) = \int_{\Omega^+} dV' \nabla \cdot h_{i,0}(r) = 0$, the Neumann-condition is viable. The field $h_i = h_{i,0} + h_{i,1}$ in Ω^+ now obeys

$$\nabla \cdot h_i = 0, \quad \nabla \times h_i = 0, \quad \hat{n} \cdot h_i = 0. \tag{8}$$

If the surface current $j_i(r) = \hat{n} \times h_i(r)$ is introduced on Γ and the magnetic field is extended by zero in Ω^-, this configuration of bounded fields and sources fulfills the Maxwell equations and jump conditions in all of space *and* obeys the PEC boundary conditions for the magnetic field. Therefore, the current $j_i(r)$ resides in the nullspace of the static MFIE operator.

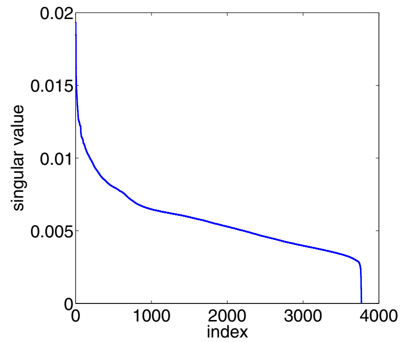
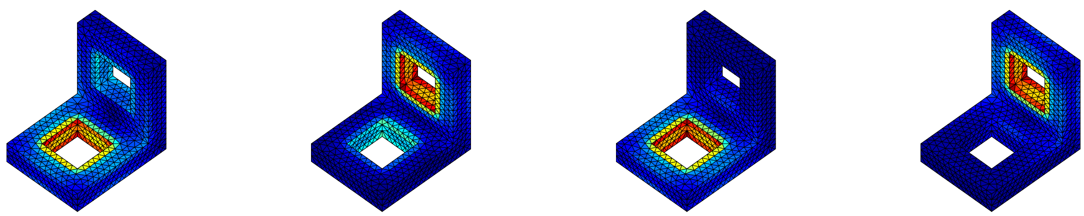


Figure 1: Singular spectrum of the discretized static MFIE operator for the 2-torus.



(a) (b) (c) (d)

Figure 2: Current intensity plot of the singular vectors in the kernel of the static outer MFIE operator (a) and (b), and their decomposition in fundamental modes (c) and (d).

Furthermore, it can be proven that the N currents $j_i(r)$ thus constructed form a basis for the nullspace of the static MFIE. More explicitly, if a given current $j(r)$ resides in the nullspace, it can be expanded as

$$j(r) = \sum_{i=1}^{N} c_i h_i(r) \tag{9}$$

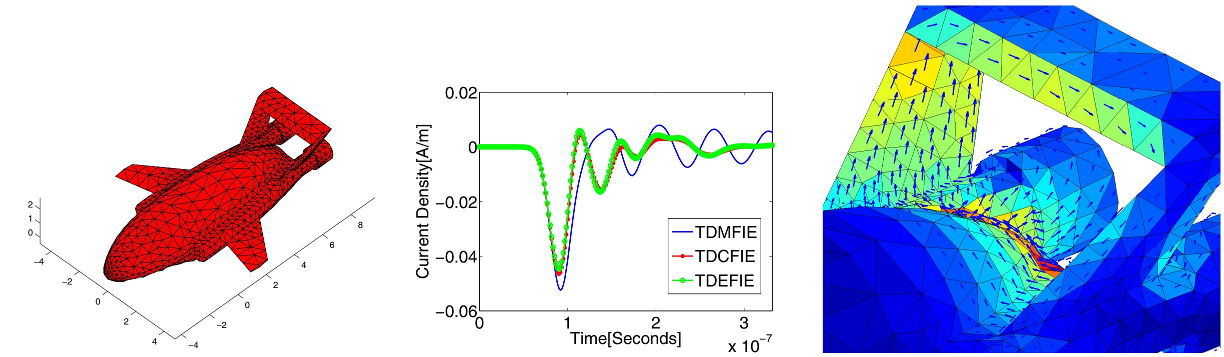
where C_i is

$$c_i = \int_{C_i} h(r) \cdot dl. \tag{10}$$

where C_i is a loop in Ω^+ that circles hole i. The analysis conducted above can be repeated to construct the nullspace of the inner static MFIE operator $-1/2 + K^s$. In this case, the curves L_i circle the holes in Ω^+ and the loops C_i circle the holes in Ω^-. The currents in the nullspace only cause non-zero fields in Ω^-. Moreover, it can be proven that the sum of the nullspaces of the outer and inner operator is direct. This means in practice that a current in the sum of the nullspace of the outer and inner static MFIE operator can be decomposed in a current that belongs to the nullspace of the outer static MFIE operator and a current that belongs to the nullspace of the inner static MFIE operator by computing the line integrals of the corresponding magnetic fields along loops circling the holes in Ω^+ and Ω^-, respectively.

4. Examples

Both the FD-MFIE-MOM and TD-MFIE-MOT system are affected by the presence of a non-trivial nullspace of the static MFIE operator. In the case of the FD-MFIE equation, the nullspace of the static operator resides approximately in the nullspace of the low-frequency operator. The FD-MFIE-MOM system thus becomes ill-conditioned. In the case of the TD-MFIE equation, the presence of the nullspace results in a constant amplitude non-physical tail superposed on the true solution.
Consider the geometry of genus 2, shown in Fig. 2. The discretized outer static MFIE operator $\mathsf{K}^s + \frac{1}{2}\mathsf{G}$ was computed. A singular value decomposition of the system matrix was performed, revealing the presence of a two-dimensional nullspace.



(a) (b) (c)

Figure 3: Mesh of a model spacecraft (a). Solution of MOT TDMFIE, TDCFIE and TDEFIE (b). Current pattern of the MFIE solution at $t = 3.32e-7$ seconds (c).

The result of this SVD is plot in Fig. 1. The currents $j'_1(r)$ and $j'_2(r)$, corresponding to the two smallest singular values, are plot in Fig. 2(a) and (b). They are linear combinations of the basis elements constructed in section 3., more precisely

$$\begin{aligned} j'_1(r) &= c_{1,1} j_1(r) + c_{1,2} j_2(r) \\ j'_2(r) &= c_{2,1} j_1(r) + c_{2,2} j_2(r) \end{aligned} \tag{11}$$

where $c_{i,j}$ is the line integral of the magnetic field caused by $j'_1(r)$ along loop C_j. These coefficients were computed and (11) was solved for the fundamental modes. These modes can be seen in Fig. 2(c) and Fig. 2(d). Next consider the mesh of a model spacecraft, shown in Fig. 3(a). This model was illuminated by a transient wave

$$H^i(r,t) = \frac{4}{T\sqrt{\pi}} \hat{x} e^{-\gamma^2} \tag{12}$$

with $\gamma = \frac{4}{T}(ct - ct_0 - \hat{x}\cdot r)$, $T = 20.0$ meter, $t_0 = 1.007\text{e-}7$ s. The TDMFIE was discretized using a time step $\Delta t = 1.67$ ns. As a comparison, the structure was also analysed using the TDEFIE and TDCFIE. As can be seen in Fig. 3(b), the TDMFIE result differs from the TDEFIE and TDCFIE results. This is indicative for the presence of a nullspace current. In Fig. 3(c), the current's stream patters is plot, confirming the nature of the current. It is mainly located in the tail area and the current loops the hole formed by the tail.

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